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First-Order Invariant Einstein–Cartan Variational Structures

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A first-order invariant Einstein–Cartan structure is a Lagrangian structure on a differential manifold defined by a generally invariant Lagrangian depending on a metric field, a connection field, and the first derivatives of these fields. Moreover, it is assumed that the metric and connection fields satisfy the so-called compatibility condition. In this paper the problem of finding all such invariant Einstein–Cartan structures is discussed. It is shown that each Lagrangian of these structures depends only on certain tensors constructed from the metric and the connection fields, which means that all the Lagrangians can be described within the framework of the classical theory of invariants. The maximal number of functionally independent Lagrangians is determined as a function of the dimension of the underlying manifold.

1. INTRODUCTION

A first-order invariant Einstein–Cartan structure is a Lagrangian structure (Trautman, 1972; Krupka and Trautman, 1974) defined by the generally invariant Lagrangian depending on a metric field, a connection field, and the first derivatives of these fields. Examples of these structures are well known from various considerations concerning the internal spin of matter as the source of the gravitational field (Trautman, 1976; Kopczynski, 1975) and from the literature on the variational principles of the general relativity theory (Rund and Lovelock, 1972; Rund, 1967).

Krupka and Trautman (Krupka and Trautman, 1974; Krupka, 1974) have shown that every rth-order invariant Lagrangian structure is uniquely determined by an L_n^r-invariant Lagrangian defined on a differential manifold endowed with an action of the differential group L_n^r. It has also become clear that all such invariant Lagrangians are, at least in theory, computable (Krupka, 1976; Krupka, to appear; Novotný, to appear). In this paper we

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apply the same method to the problem of classifying the first-order invariant Einstein–Cartan structures.

Each first-order Lagrangian depending on a metric and a connection field is defined on an open subset of the manifold $T_n{}^1Q$ of one-jets with source at the origin 0 of the n-dimensional real Euclidean space R^n and target in the manifold $Q = (R^{n*} \odot R^{n*}) \times R^{n^3}$, where $R^{n*} \odot R^{n*}$ denotes the space of second-order, symmetric covariant tensors on R^n and the factor R^{n^3} corresponds to the elements of the connection. The Lie group $L_n{}^3$ of all invertible three-jets with source and target at $0 \in R^n$ acts on the manifold $T_n{}^1Q$ in a well-known manner. More detailed general information on this action and the theory of jets can be found in (Krupka, 1974 and Ehresmann, 1953). The corresponding $L_n{}^3$-invariant functions are in fact identical to the Lagrangians of the considered Lagrangian structures.

The second section of this paper is devoted to the definition of suitable local coordinates on the manifold $T_n{}^1Q$ and to a coordinate description of the action of the group $L_n{}^3$ on $T_n{}^1Q$.

In the third section we consider the Lie algebra $l_n{}^3(T_n{}^1Q)$ of fundamental vector fields on $T_n{}^1Q$ defined by the group $L_n{}^3$. We find a system of vector fields generating the Lie algebra $l_n{}^3(T_n{}^1Q)$ and characterize the rank of $l_n{}^3(T_n{}^1Q)$ at its maximal points. The rank of a Lie algebra of fundamental vector fields is the important characteristic that determines the maximal number of functionally independent integral functions of this algebra (Hermann, 1968). We reach the conclusion that the problem of finding all $L_n{}^3$-invariant functions on $T_n{}^1Q$ can be reduced to the problem of finding all GL_n-invariant functions depending on some tensors, i.e., to the problem of the classical theory of GL_n-invariants. This problem can be solved with the help of the algebraic theory of invariants (Gurevič, 1948; Dieudonné and Carrell, 1971). The second result obtained is the maximal number of functionally independent first-order invariant Lagrangians depending on a metric and a connection.

In the fourth section, these results are applied to the first-order invariant Einstein–Cartan structures. In particular, the maximal number of functionally independent Lagrangians of these structures on n-dimensional manifolds is determined which is equal to 0, 9, 57, 194 for $n = 1, 2, 3, 4$, respectively.

The last section is devoted to the study of some special classes of the first-order invariant Lagrangians depending on a metric and a connection. In a special case of Lagrangians, our results are in agreement with those of Rund (1967).

2. FUNDAMENTAL GEOMETRICAL STRUCTURES

Let us consider the manifolds Q and $T_n{}^1Q$ introduced in the first section and define in terms of some coordinates the standard action of the group $L_n{}^3$ on the manifold $T_n{}^1Q$ [for the generalities, see (Krupka, 1974)].

Let g_{ij}, Γ^i_{jk} be the canonical coordinates on Q, and let g_{ij}, Γ^i_{jk}, $g_{ij,k}$, $\Gamma^i_{jk,l}$ be the associated canonical coordinates on the manifold $T_n{}^1Q$. In accordance with the general theory, each first-order generally invariant Lagrangian depending on a metric field and a connection field is a function of the variables g_{ij}, Γ^i_{jk}, $g_{ij,k}$, $\Gamma^i_{jk,l}$.

Denote by $a_j{}^i$, a^i_{jk}, a^i_{jkl} the canonical coordinates on $L_n{}^3$. The group $L_n{}^2$ acts on the manifold Q by

$$\begin{aligned} \bar{g}_{ik} &= b_i{}^m b_k{}^n g_{mn} \\ \bar{\Gamma}^i_{kl} &= \Gamma^m_{np} a_m{}^i b_k{}^n b_l{}^p - b_k{}^r b_l{}^s a^i_{rs} \end{aligned} \tag{2.1}$$

where $b_k{}^j$ is defined by $a_j{}^i b_k{}^j = \delta_k{}^i$ and $\delta_k{}^i$ is the Kronecker symbol. The group action of $L_n{}^3$ on $T_n{}^1Q$, associated with this action, is given by (2.1) and by

$$\begin{aligned} \bar{g}_{ik,l} &= b_i{}^m b_k{}^n b_l{}^p g_{mn,p} - b_a{}^m b_l{}^p b_i{}^c b_k{}^n (a^a_{bc} g_{mn} + a^a_{bn} g_{cm}) \\ \bar{\Gamma}^i_{kl,q} &= \Gamma^m_{np,s} a_m{}^i b_q{}^s b_k{}^n b_l{}^p + \Gamma^m_{np} a^i_{ms} b_q{}^s b_k{}^n b_l{}^p \\ &\quad - \Gamma^m_{np} a_m{}^i b_q{}^c a^b_{cd} (b_l{}^p b_b{}^n b_k{}^d + b_b{}^p b_k{}^n b_l{}^d) \\ &\quad + b_l{}^f b_k{}^d b_q{}^g b_h{}^b (a^i_{db} a^h_{gf} + a^i_{bf} a^h_{gd}) - b_l{}^f b_k{}^d b_q{}^g a^a_{g\,df} \end{aligned} \tag{2.2}$$

As usual, if a system of numbers g_{ij}, Γ^i_{jk}, $g_{ij,k}$, $\Gamma^i_{jk,l}$ denotes the coordinates of a point $j_0{}^1 f \in T_n{}^1Q$ and if $a_j{}^i$, a^i_{jk}, a^i_{jkl} are the coordinates of an element $j_0{}^3\alpha \in L_n{}^3$, then $\bar{g}_{ij}$, $\bar{\Gamma}^i_{jk}$, $\bar{g}_{ij,k}$, $\bar{\Gamma}^i_{jk,l}$ is the system of the coordinates of the transformed point $j_0{}^3\alpha \cdot j_0{}^1 f \in T_n{}^1Q$.

Let us consider a subset $W \subset T_n{}^1Q$ formed by the points, where $\det(g_{ij}) \neq 0$, and let us introduce the functions g^{ij} on W by the relation $g^{ij} g_{jk} = \delta_k{}^i$. On W a new coordinate system can be introduced by the relations

$$\begin{aligned} g^{ij} &= g^{ij} \\ W^{ij}_k &= g^{ij}{}_{;k} = g^{ij}{}_{,k} + \Gamma^i_{mk} g^{mj} + \Gamma^j_{mk} g^{im} \\ T^i_{jk} &= \Gamma^i_{jk} - \Gamma^i_{kj} \\ S^i_{jk} &= \Gamma^i_{jk} + \Gamma^i_{kj} \\ R^l_{kij} &= \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik} \\ V^l_{jki} &= T^l_{jk;i} = \Gamma^l_{jk,i} - \Gamma^l_{kj,i} + (\Gamma^s_{jk} - \Gamma^s_{kj})\Gamma^l_{si} \\ &\qquad - (\Gamma^l_{sk} - \Gamma^l_{ks})\Gamma^s_{ji} - (\Gamma^l_{js} - \Gamma^l_{sj})\Gamma^s_{ik} \\ U^l_{ijk} &= \tfrac{1}{6}(\Gamma^l_{ij,k} + \Gamma^l_{ik,j} + \Gamma^l_{ji,k} + \Gamma^l_{jk,i} + \Gamma^l_{ki,j} + \Gamma^l_{kj,i}) \end{aligned} \tag{2.3}$$

It can easily be shown that the inverse transformation is

$$\begin{aligned} g^{ij} &= g^{ij} \\ \Gamma^i_{jk} &= \tfrac{1}{2}(T^i_{jk} + S^i_{jk}) \\ g^{ij}{}_{,k} &= W^{ij}_k - \tfrac{1}{2} g^{mj}(T^i_{mk} + S^i_{mk}) - \tfrac{1}{2} g^{im}(T^j_{mk} + S^j_{mk}) \\ \Gamma^l_{jk,i} &= U^l_{ijk} + \tfrac{1}{3}(V^l_{jki} + V^l_{ikj}) + \tfrac{1}{6}(R^l_{ijk} + R^l_{jik} + 3R^l_{kij}) \\ &\quad + \tfrac{1}{12}(2T^l_{js}T^s_{ik} - T^l_{is}T^s_{jk} - 3T^s_{jk}S^l_{is} + 3T^l_{js}S^s_{ik} \\ &\quad - 3T^l_{ks}S^s_{ij} - 2S^l_{is}S^s_{jk} + S^l_{js}S^s_{ik} + S^l_{ks}S^s_{ij}) \end{aligned} \tag{2.4}$$

and that the following identities hold.

$$R^h_{ikl} + R^h_{lik} + R^h_{kli} = V^h_{ikl} + V^h_{lik} + V^h_{kli} - T^j_{ik}T^h_{lj} - T^j_{li}T^h_{kj} - T^j_{kl}T^h_{ij}$$

In these coordinates, the action (2.1), (2.2) of L_n^3 on T_n^1Q is expressed by

$$\begin{aligned}
\bar{g}^{ij} &= a_m{}^i a_n{}^j g^{mn} \\
\bar{W}^{ij}_k &= a_m{}^i a_n{}^j b_k{}^p W^{mn}_p \\
\bar{T}^i_{jk} &= a_m{}^i b_j{}^n b_k{}^p T^m_{np} \\
\bar{S}^i_{jk} &= a_m{}^i b_j{}^n b_k{}^p S^m_{np} - 2b_j{}^q b_k{}^s a^i_{qs} \\
\bar{R}^i_{jkl} &= a_m{}^i b_j{}^n b_k{}^p b_l{}^q R^m_{npq} \\
\bar{V}^i_{jkl} &= a_m{}^i b_j{}^n b_k{}^p b_l{}^q V^m_{npq} \\
\bar{U}^i_{jkl} &= \tfrac{1}{6}[a_m{}^i b_l{}^s b_j{}^n b_k{}^p U^m_{nps} + 2b_l{}^s b_j{}^n b_k{}^p (S^m_{np}a^i_{ms} + S^m_{ns}a^i_{mp} + S^m_{sp}a^i_{mn}) \\
&\quad - 4b_k{}^p b_b{}^n b_j{}^d b_l{}^c a_m{}^i (S^m_{np}a^b_{cd} + S^m_{nd}a^b_{cp} + S^m_{nc}a^b_{pd}) \\
&\quad + 4b_k{}^f b_j{}^d b_l{}^g b_h{}^b (a^i_{db}a^h_{gf} + a^i_{bf}a^h_{gd} + a^i_{bg}a^h_{fd}) - 6b_k{}^f b_j{}^d b_l{}^g a^i_{gdf}]
\end{aligned} \tag{2.5}$$

These are the desired formulas.

3. INVARIANT LAGRANGIANS

Let G be a Lie group and M a differential manifold; let the map $G \times M \ni (g, x) \to g \cdot x \in M$ define an action of G on M. Recall that a real function f defined on M is said to be G-invariant (or just invariant) if $f(g \cdot x) = f(x)$ for all $x \in M$ and $g \in G$.

Our problem is to characterize all L_n^3-invariant functions defined on the manifold T_n^1Q. The general theory tells us that each L_n^3-invariant function f satisfies the complete system of differential identities

$$\Xi_i{}^j(f) = 0, \qquad \Xi_i^{jk}(f) = 0, \qquad \Xi_i^{jkl}(f) = 0 \tag{3.1}$$

where $\Xi_i{}^j$, Ξ_i^{jk}, Ξ_i^{jkl} are the fundamental vector fields on T_n^1Q, defined by the action (2.5) of the group L_n^3 on T_n^1Q. Equations (3.1) form a system of linear, homogeneous, first-order partial differential equations for the function f. The classical Frobenius theorem (Hermann, 1968) ensures the existence of the nontrivial solutions of (3.1) and enables us to determine the maximal number of functionally independent solutions.

Let us construct the vector fields $\Xi_i{}^j$, Ξ_i^{jk}, Ξ_i^{jkl} (3.1). By a standard differentiation procedure we easily obtain from (2.5) that in the local coordinates g^{ij}, W_k^{ij}, T_{jk}^i, S_{jk}^i, R_{ijk}^l, V_{ijk}^l, U_{ijk}^l on W, it holds

$$\begin{aligned}
\Xi_\alpha{}^\beta = {} & 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + 2W_k^{\beta j}\frac{\partial}{\partial W_k^{\alpha j}} - W_\alpha^{ij}\frac{\partial}{\partial W_\beta^{ij}} + T_{jk}^\beta\frac{\partial}{\partial T_{jk}^\alpha} \\
& - 2T_{\alpha k}^i\frac{\partial}{\partial T_{\beta k}^i} + R_{jkl}^\beta\frac{\partial}{\partial R_{jkl}^\alpha} - R_{\alpha kl}^i\frac{\partial}{\partial R_{\beta kl}^i} - 2R_{j\alpha l}^i\frac{\partial}{\partial R_{j\beta l}^i} \\
& + V_{jkl}^\beta\frac{\partial}{\partial V_{jkl}^\alpha} - 2V_{\alpha kl}^i\frac{\partial}{\partial V_{\beta kl}^i} - V_{jk\alpha}^i\frac{\partial}{\partial V_{jk\beta}^i}
\end{aligned} \tag{3.2}$$

$$\Xi_\alpha^{\beta\gamma} = \frac{\partial}{\partial S_{\beta\gamma}^\alpha}$$

$$\Xi_\alpha^{\beta\gamma\delta} = \frac{\partial}{\partial U_{\beta\gamma\delta}^\alpha}$$

These vector fields obviously span the Lie algebra $l_n^3(T_n^1Q)$ on W.

We can now determine the rank of $l_n^3(T_n^1Q)$ at its maximal points, i.e., at points of the manifold T_n^1Q where the number of linearly independent fundamental vector fields is maximal. To do this, we define new local coordinates R_{ijkl} on W by the relation

$$R_{ijkl} = g_{im}R_{jkl}^m$$

and put

$$\Xi_{\alpha\gamma} = g_{\beta\gamma}\Xi_\alpha{}^\beta$$

$$\Xi_{\alpha\gamma}^+ = \tfrac{1}{2}(\Xi_{\alpha\gamma} + \Xi_{\gamma\alpha}), \qquad \Xi_{\alpha\gamma}^- = \tfrac{1}{2}(\Xi_{\alpha\gamma} - \Xi_{\gamma\alpha})$$

The system (3.2) of fundamental vector fields on W is thus equivalent to the system

$$\begin{aligned}
\Xi_{\alpha\gamma}^+ = {} & 2\frac{\partial}{\partial g^{\alpha\gamma}} + \tfrac{1}{2}(2g_{k\gamma}W_\beta^{kj}\delta_\alpha{}^i + 2g_{k\alpha}W_\beta^{kj}\delta_\gamma{}^i - g_{\beta\gamma}W_\alpha^{ij} - g_{\beta\alpha}W_\gamma^{ij})\frac{\partial}{\partial W_\beta^{ij}} \\
& + \tfrac{1}{2}(g_{j\gamma}T_{\beta k}^j\delta_\alpha{}^i + g_{j\alpha}T_{\beta k}^j\delta_\gamma{}^i - 2g_{\beta\gamma}T_{\alpha k}^i - 2g_{\beta\alpha}T_{\gamma k}^i)\frac{\partial}{\partial T_{\beta k}^i} \\
& - \tfrac{1}{2}(g_{i\gamma}R_{\alpha jk\beta} + g_{i\alpha}R_{\gamma jk\beta} + g_{j\gamma}R_{i\alpha k\beta} + g_{j\alpha}R_{i\gamma k\beta} \\
& \quad + 2g_{\beta\gamma}R_{ijk\alpha} + 2g_{\beta\alpha}R_{ijk\gamma})\frac{\partial}{\partial R_{ijk\beta}} \\
& + \tfrac{1}{2}(g_{l\gamma}V_{jk\beta}^l\delta_\alpha{}^i + g_{j\alpha}V_{jk\beta}^l\delta_\gamma{}^i - 2g_{j\gamma}V_{\alpha k\beta}^i - 2g_{j\alpha}V_{\gamma k\beta}^i \\
& \quad - g_{\beta\gamma}V_{jk\alpha}^i - g_{\beta\alpha}V_{jk\gamma}^i)\frac{\partial}{\partial V_{jk\beta}^i}
\end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\Xi_{\alpha\gamma}^{-} = {} & \tfrac{1}{2}(2g_{k\gamma}W^{kj}_{\beta}\delta_{\alpha}{}^{i} - 2g_{k\alpha}W^{kj}_{\beta}\delta_{\gamma}{}^{i} - g_{\beta\gamma}W^{ij}_{\alpha} + g_{\beta\alpha}W^{ij}_{\gamma})\frac{\partial}{\partial W^{ij}_{\beta}} \\
& + \tfrac{1}{2}(g_{j\gamma}T^{j}_{\beta k}\delta_{\alpha}{}^{i} - g_{j\alpha}T^{j}_{\beta k}\delta_{\gamma}{}^{i} - 2g_{\beta\gamma}T^{i}_{\alpha k} + 2g_{\beta\alpha}T^{i}_{\gamma k})\frac{\partial}{\partial T^{i}_{\beta k}} \\
& - \tfrac{1}{2}(g_{i\gamma}R_{\alpha jk\beta} - g_{i\alpha}R_{\gamma jk\beta} + g_{j\gamma}R_{i\alpha k\beta} - g_{j\alpha}R_{i\gamma k\beta} \\
& \quad + 2g_{\beta\gamma}R_{ijk\alpha} - 2g_{\beta\alpha}R_{ijk\gamma})\frac{\partial}{\partial R_{ijk\beta}} \\
& + \tfrac{1}{2}(g_{l\gamma}V^{l}_{jk\beta}\delta_{\alpha}{}^{i} - g_{j\alpha}V^{l}_{jk\beta}\delta_{\gamma}{}^{i} - 2g_{j\gamma}V^{i}_{\alpha k\beta} + 2g_{j\alpha}V^{i}_{\gamma k\beta} \\
& \quad - g_{\beta\gamma}V^{i}_{jk\alpha} + g_{\beta\alpha}V^{i}_{jk\gamma})\frac{\partial}{\partial V^{i}_{jk\beta}} \qquad (3.3)
\end{aligned}
$$

$$\Xi_{\alpha}^{\beta\gamma} = \frac{\partial}{\partial S^{\alpha}_{\beta\gamma}}$$

$$\Xi_{\alpha}^{\beta\gamma\delta} = \frac{\partial}{\partial U^{\alpha}_{\beta\gamma\delta}}$$

The rank of this system is defined as the rank of the matrix formed by the coefficients in (3.3) at the base vector fields

$$\frac{\partial}{\partial g^{ij}},\quad \frac{\partial}{\partial W^{ij}_{k}},\quad \frac{\partial}{\partial T^{i}_{jk}},\quad \frac{\partial}{\partial R_{ijkl}},\quad \frac{\partial}{\partial V^{i}_{jkl}},\quad \frac{\partial}{\partial S^{i}_{jk}},\quad \frac{\partial}{\partial U^{i}_{jkl}}$$

The form of the vector fields (3.3) shows that the rank of this matrix at its maximal points is given by

$$r_n = r_n' + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}n^2(n+1) + n\binom{n+2}{3}$$

where r_n' is the rank of the matrix

$$
\begin{array}{c|c:|}
\Xi_{\alpha\gamma}^{-} & & \\
\downarrow\ \alpha < \gamma & \Delta_n & \\
\end{array}
\qquad
\begin{array}{c}
\dfrac{\partial}{\partial R_{ijji}},\quad i<j \\
\longrightarrow
\end{array}
$$

Similarly as in (Krupka, 1976) one can show that the determinant $\det(\Delta_n) \neq 0$ and thus at some point of $T_n{}^1Q$, $r_n' = (\tfrac{1}{2})\cdot n(n-1)$. The rank of the system (3.3) or (3.2) of fundamental vector fields, i.e., the rank of the Lie algebra $l_n{}^3(T_n{}^1Q)$ at its maximal points, is thus equal to

$$r_n = \tfrac{1}{6}n^2(n^2 + 6n + 11)$$

According to the general theory of vector field systems (Hermann, 1968), the basis of integral functions of the Lie algebra $l_n^3(T_n^1 Q)$ includes exactly

$$M_n = \dim T_n^1 Q - r_n = \tfrac{1}{6}n(5n^3 + 3n^2 - 5n + 3)$$

functions.

Our results can be summarized as follows.

Theorem. Each generally invariant Lagrangian L depending on a metric field, a connection field, and the first derivatives of these fields satisfies the complete system of differential identities

$$\Xi_\alpha{}^\beta(L) = 0, \qquad \Xi_\alpha^{\beta\gamma}(L) = 0, \qquad \Xi_\alpha^{\beta\gamma\delta}(L) = 0 \tag{3.4}$$

where $\Xi_\alpha{}^\beta$, $\Xi_\alpha^{\beta\gamma}$, $\Xi_\alpha^{\beta\gamma\delta}$ are given by (3.2). There exist at most

$$M_n = \tfrac{1}{6}n(5n^3 + 3n^2 - 5n + 3)$$

functionally independent generally invariant Lagrangians.

Note that the form of the vector fields (3.2) implies that each Lagrangian of the considered class depends only on the coordinates g^{ij}, W_k^{ij}, T_{jk}^i, R_{ijk}^l, V_{ijk}^l. These coordinates transform, under the transformations from L_n^3 as tensors of GL_n, by (2.5). We can therefore conclude that the problem of finding all L_n^3-invariant Lagrangians on $W \subset T_n^1 Q$ is reduced to a problem of finding all invariants of the tensors g^{ij}, W_k^{ij}, T_{jk}^i, R_{ijk}^l, V_{ijk}^l, i.e., to a problem of the classical invariant theory (Gurevič, 1948; Dieudonné and Carrell, 1971).

4. EINSTEIN–CARTAN STRUCTURES

Let us discuss the possible invariant Lagrangians for the Einstein–Cartan theories of gravitation. The underlying structure for these theories is a four-dimensional differential manifold with a metric tensor field and a linear connection compatible with the metric. The connection is not symmetric, in general. Recall that a linear connection with the components Γ_{jk}^i is said to be compatible with the metric with the components g_{ij} if

$$g_{ij;k} = g_{ij,k} - \Gamma_{ik}^s g_{sj} - \Gamma_{jk}^s g_{is} = 0 \tag{4.1}$$

i.e., if the covariant derivative of g_{ij} by Γ_{jk}^i vanishes.

From the point of view of the variational theory the condition (4.1) means that a first-order Lagrangian defining an Einstein–Cartan structure is independent of the coordinates $g_{ij,k}$ or, in our coordinates (2.3), (2.4), independent of W_k^{ij}. Each such generally invariant Lagrangian is thus defined on a subset of the manifold $P = (R^{n*} \odot R^{n*}) \times T_n^1 R^{n^3}$, where $T_n^1 R^{n^3}$ is the manifold of one-jets with source at $0 \in R^n$ and target in R^{n^3}. The natural action of L_n^3 on P is immediately seen from (2.5).

The fundamental vector fields on P generated by this action can be obtained easily. In terms of the coordinates g^{ij}, T^i_{jk}, S^i_{jk}, R^i_{jkl}, V^i_{jkl}, U^i_{jkl} on P these vector fields are given by

$$\begin{aligned}\Xi_\alpha{}^\beta &= 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + T^\beta_{jk}\frac{\partial}{\partial T^\alpha_{jk}} - 2T^i_{\alpha k}\frac{\partial}{\partial T^i_{\beta k}} + R^\beta_{jkl}\frac{\partial}{\partial R^\alpha_{jkl}} - R^i_{\alpha kl}\frac{\partial}{\partial R^i_{\beta kl}} \\ &\quad - 2R^i_{j\alpha l}\frac{\partial}{\partial R^i_{j\beta l}} + V^\beta_{jkl}\frac{\partial}{\partial V^\alpha_{jkl}} - 2V^i_{\alpha kl}\frac{\partial}{\partial V^i_{\beta kl}} - V^i_{jk\alpha}\frac{\partial}{\partial V^i_{jk\beta}} \\ \Xi_\alpha^{\beta\gamma} &= \frac{\partial}{\partial S^\alpha_{\beta\gamma}} \\ \Xi_\alpha^{\beta\gamma\delta} &= \frac{\partial}{\partial U^\alpha_{\beta\gamma\delta}}\end{aligned} \tag{4.2}$$

The rank of the Lie algebra $l_n{}^3P$ is given by

$$r_n = \tfrac{1}{6}n^2(n^2 + 6n + 11)$$

This can be checked in the same way as in the preceding case.

We have thus arrived at the following conclusions.

Theorem. Each generally invariant Lagrangian defining an Einstein–Cartan structure satisfies the complete system of differential identities (3.4), where $\Xi_\alpha{}^\beta$, $\Xi_\alpha^{\beta\gamma}$, $\Xi_\alpha^{\beta\gamma\gamma}$ are given by (4.2). There exist at most

$$M_n = \dim P - r_n = \tfrac{1}{6}n(5n^3 - 8n + 3)$$

functionally independent generally invariant Lagrangians.

In particular, each first-order $L_n{}^3$-invariant Lagrangian of an Einstein–Cartan structure depends only on the metric tensor, the curvature and torsion tensor, and the covariant derivative of the torsion tensor.

Consider for example the case $n = 2$. Then $M_n = 9$ and it should not be so difficult to obtain a basis of $L_n{}^3$-invariant functions without a computer. In fact, it is directly proved that a basis of the corresponding generally invariant Lagrangians can be taken as

$$\begin{aligned}L_1 &= g^{ij}R^m_{imj} \\ L_2 &= g^{ij}g^{kl}R^m_{kmi}R^n_{lnj} \\ L_3 &= g^{ij}g^{kl}R^m_{kmi}R^n_{jnl} \\ L_4 &= g^{ij}g^{kl}V^m_{mik}R^n_{njl} \\ L_5 &= g^{ij}V^m_{mij} \\ L_6 &= g^{ij}g^{kl}V^m_{mik}V^n_{njl} \\ L_7 &= g^{ij}g^{kl}V^m_{mik}V^n_{nlj} \\ L_8 &= g^{ij}g^{kl}R^m_{ikm}V^n_{jln} \\ L_9 &= g^{ij}T^k_{li}T^l_{kj}\end{aligned}$$

5. SPECIAL CLASSES OF LAGRANGIANS

In this section we discuss some special classes of the first-order generally invariant Lagrangians depending on a metric field and a connection field.

(a) $L(g_{ij}, \Gamma^i_{jk}, \Gamma^i_{jk,l})$, $\Gamma^i_{jk} = \Gamma^i_{kj}$. Putting $T^i_{jk} = 0$ and $V^i_{jkl} = 0$ we obtain from (4.2) that the corresponding fundamental vector fields are of the form

$$\begin{aligned}
\Xi_\alpha{}^\beta &= 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + R^\beta_{jkl}\frac{\partial}{\partial R^\alpha_{jkl}} - R^i_{\alpha kl}\frac{\partial}{\partial R^i_{\beta kl}} - 2R^i_{j\alpha l}\frac{\partial}{\partial R^i_{j\beta l}} \\
\Xi^{\beta\gamma}_\alpha &= \frac{\partial}{\partial S^\alpha_{\beta\gamma}} \qquad\qquad (5.1) \\
\Xi^{\beta\gamma\delta}_\alpha &= \frac{\partial}{\partial U^\alpha_{\beta\gamma\delta}}
\end{aligned}$$

The rank of the Lie algebra $l_n{}^3[(R^{n*} \odot R^{n*}) \times R^m]$, $m = (\frac{1}{2})\cdot n^2(n+1)$, is given by

$$r_n = \tfrac{1}{6}n^2(n^2 + 6n + 11)$$

and there exist at most

$$M_n = \tfrac{1}{6}n(2n^3 - 5n + 3)$$

functionally independent Lagrangians.

We note that this class of Lagrangians was investigated by Rund. It can be seen by a direct calculation in the canonical coordinates that our fundamental vector-field system (5.1) is equivalent to his first, second, and third invariance identities (Rund, 1967).

(b) $L(g_{ij}, \Gamma^i_{jk}, \Gamma^j_{jk,l})$, where Γ^i_{jk} is a metric connection. Consider the case of the metric connection. Then

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}) \tag{5.2}$$

and $\Gamma^i_{jk,l}$ is expressed by means of the coordinates $g_{ij}, g_{ij,k}, g_{ij,kl}$. This means that the corresponding generally invariant Lagrangians are defined on the manifold $T_n{}^2(R^{n*} \odot R^{n*})$ of two-jets with source at $0 \in R^n$ and target in $R^{n*} \odot R^{n*}$. In (2.3) put $W^{ij}_k = 0$, $T^i_{jk\delta} = 0$, $V^{i\delta}_{jkl} = 0$ and introduce new local coordinates R_{ijkl}, U_{ijkl}, Δ_{ijk} by

$$\begin{aligned}
R_{ijkl} &= g_{im}R^m_{jkl} \\
U_{ijkl} &= g_{im}U^m_{jkl} \\
\Delta_{ijk} &= \tfrac{1}{2}g_{im}S^m_{jk}
\end{aligned}$$

On comparing these coordinates with those on $T_n{}^2(R^{n*} \odot R^{n*})$ introduced by Krupka (to appear), we can conclude that the converse is also true: Each second-order generally invariant Lagrangian $L(g_{ij}, g_{ij,k}, g_{ij,kl})$ defines a generally invariant Lagrangian $L(g_{ij}, \Gamma^i_{jk}, \Gamma^i_{jk,l})$, where (5.2) holds. The generally invariant Lagrangians $L(g_{ij}, g_{ij,k}, g_{ij,kl})$ have been characterized by Krupka (to appear; 1976).

(c) $L(g_{ij}, g_{ij,k}, \Gamma^i_{jk})$. The Lagrangians of this type are defined on the manifold $T_n{}^1(R^{n*} \odot R^{n*}) \times R^{n^3}$. Each of these Lagrangians satisfies the system of differential identities

$$\Xi_\alpha{}^\beta(L) = 0, \qquad \Xi_\alpha^{\beta\gamma}(L) = 0 \tag{5.3}$$

where

$$\begin{aligned} \Xi_\alpha{}^\beta &= 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + 2W_k^{\beta j}\frac{\partial}{\partial W_k^{\alpha j}} - W_\alpha^{ij}\frac{\partial}{\partial W_\beta^{ij}} + T^\beta_{jk}\frac{\partial}{\partial T^\alpha_{jk}} - 2T^i_{\alpha k}\frac{\partial}{\partial T^i_{\beta k}} \\ \Xi_\alpha^{\beta\gamma} &= \frac{\partial}{\partial S^\alpha_{\beta\gamma}} \end{aligned} \tag{5.4}$$

and the local coordinates g^{ij}, W_k^{ij}, T^i_{jk}, S^i_{jk} are defined by the first four relations of (2.3).

To determine the rank of this system, we introduce new local coordinates

$$Q_{ijk} = g_{im}g_{jn}W_k^{mn}$$

and put

$$\Xi_{\alpha\gamma} = g_{\beta\gamma}\Xi_\alpha{}^\beta$$

$$\Xi^+_{\alpha\gamma} = \tfrac{1}{2}(\Xi_{\alpha\gamma} + \Xi_{\gamma\alpha}), \qquad \Xi^-_{\alpha\gamma} = \tfrac{1}{2}(\Xi_{\alpha\gamma} - \Xi_{\gamma\alpha})$$

As before, the rank of the vector-field system $\Xi^+_{\alpha\gamma}$, $\Xi^-_{\alpha\gamma}$, $\Xi_\alpha^{\beta\gamma}$ is given by

$$r_n = r'_n + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}n^2(n+1)$$

where r'_n is the rank of the matrix

$$\begin{array}{c} \begin{array}{c} \Xi^-_{\alpha\gamma} \\ \alpha < \gamma \end{array}\Big\downarrow \left|\begin{array}{c:c} \quad\Delta_n\quad & \quad \end{array}\right| \\ \xrightarrow{\dfrac{\partial}{\partial Q_{ijn}},\quad i<j} \end{array}$$

Since $\det \Delta_n$ does not vanish identically, we have at some points $r'_n = (\tfrac{1}{2})n(n-1)$. Consequently, there exist at most

$$M_n = \tfrac{1}{2}n(2n^2 - n + 1)$$

functionally independent generally invariant Lagrangians of the considered class.

In the case of the symmetric connection, $T^i_{jk} = 0$ and the system (5.4) is reduced to

$$\Xi_\alpha{}^\beta = 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + 2W^{\beta j}_k\frac{\partial}{\partial W^{\alpha j}_k} - W^{ij}_\alpha\frac{\partial}{\partial W^{ij}_\beta}$$

$$\Xi^{\beta\gamma}_\alpha = \frac{\partial}{\partial S^\alpha_{\beta\gamma}}$$

The rank of this system remains unchanged and the maximal number of functionally independent invariants is given by

$$M_n = \tfrac{1}{2}n(n^2 + 1)$$

(d) $L(g_{ij}, \Gamma^i_{jk})$. The Lagrangians of this class are defined on the manifold $(R^{n^*} \odot R^{n^*}) \times R^{n^3}$. Each of these Lagrangians satisfies (5.3), where

$$\begin{aligned}\Xi_\alpha{}^\beta &= 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}} + T^\beta_{jk}\frac{\partial}{\partial T^\alpha_{jk}} - 2T^i_{\alpha k}\frac{\partial}{\partial T^i_{\beta k}} \\ \Xi^{\beta\gamma}_\alpha &= \frac{\partial}{\partial S^\alpha_{\beta\gamma}}\end{aligned} \tag{5.5}$$

and the local coordinates g^{ij}, T^i_{jk}, S^i_{jk} are defined by the first, third, and fourth formulas of (2.3).

To show the independence of this system of differential operators, we introduce the local coordinates

$$P_{ijk} = g_{im}T^m_{jk}$$

and proceed in the same way as in the preceding case. Since the rank of the system (5.5) is equal to

$$r_n = \tfrac{1}{2}n^2(n + 3)$$

we can assert that there exist at most

$$M_n = \tfrac{1}{2}n(n - 1)^2$$

functionally independent generally invariant Lagrangians of the considered class.

In the case of the symmetric connection the system (5.5) is reduced to

$$\Xi_\alpha{}^\beta = 2g^{i\beta}\frac{\partial}{\partial g^{i\alpha}}$$

$$\Xi^{\beta\gamma}_\alpha = \frac{\partial}{\partial S^\alpha_{\beta\gamma}}$$

Putting

$$\Xi_{\alpha\gamma} = \tfrac{1}{2}g_{\beta\gamma}\Xi_\alpha{}^\beta = \frac{\partial}{\partial g^{\alpha\gamma}}$$

we see that there exist no nontrivial Lagrangians depending on g_{ij} and symmetric connection Γ^i_{jk}. This result agrees with (Rund, 1967).

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